Energy cost of entanglement extraction from quantum fields

QISS HKU Workshop 2020

Hong Kong

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[arXiv:1904.06246, Quantum 3, 165 (2019)]

Motivation

ENTANGLEMENT is a hallmark of quantum theory and a resource for quantum applications:

- Super-dense coding [Bennett, Wiesner '92]
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Key question of this talk:

What is the minimal price in energy ΔE to extract ΔS of entanglement?

Our setup, protocol and results

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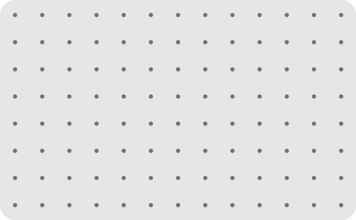
Bosonic and fermionic field theories with quadratic Hamiltonian and Gaussian ground state $|\psi\rangle$:

$$\hat{H} = \int d^3x \frac{1}{2} h_{ab} \hat{\xi}^a(x) \hat{\xi}^b(x) \qquad \text{with} \qquad \hat{\xi}^a(x) \text{ representing bosonic or fermionic fields}$$

Examples: Klein-Gordon field, Dirac field in Minkoski space or curved backgrounds.



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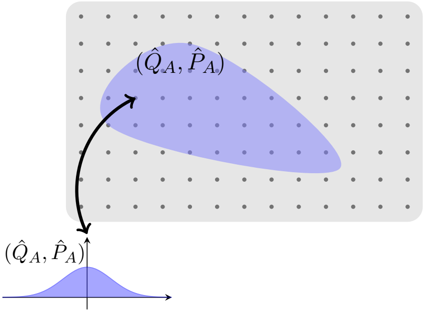

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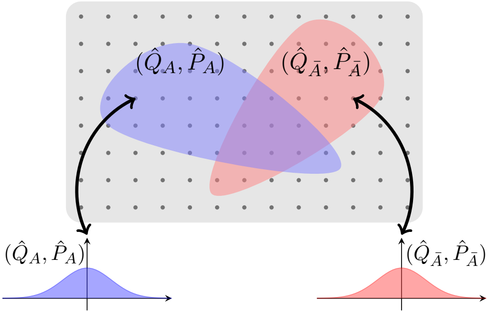

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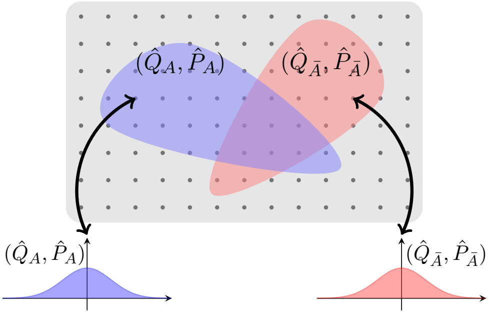

$\Rightarrow$ A and $\bar{A}$ are entangled, but together they are in a pure state $|\psi_{A\bar{A}}\rangle$!

Our protocol

Entanglement extraction:

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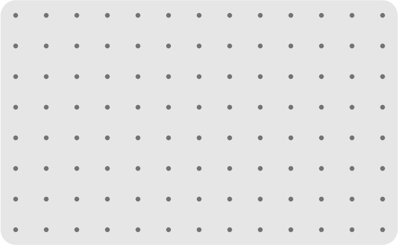
Field theory model



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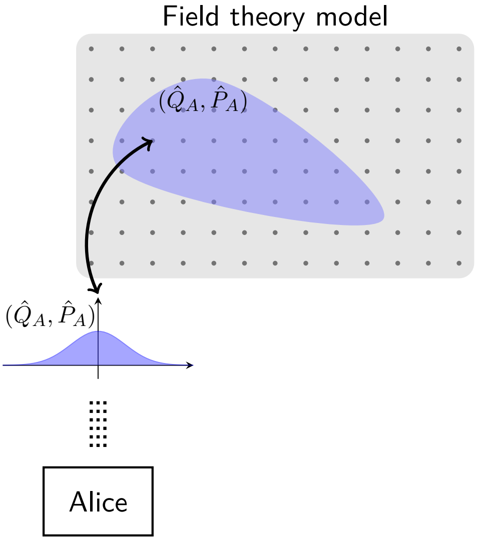

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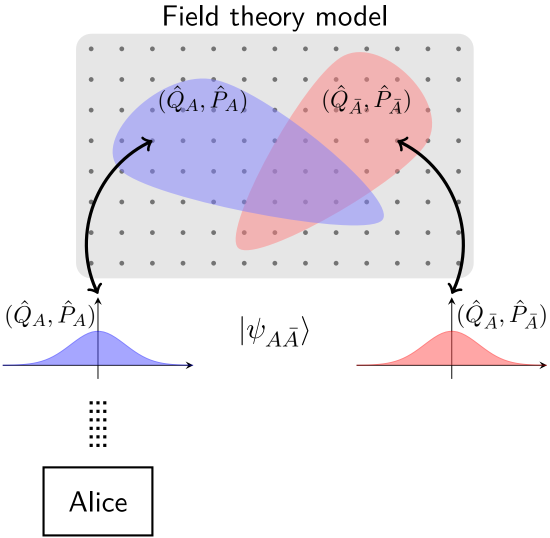

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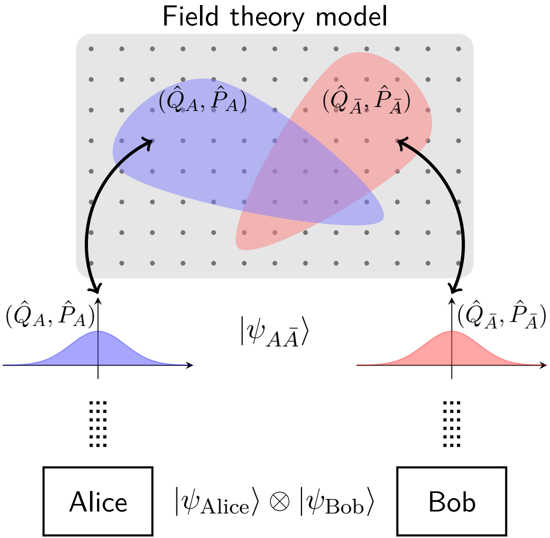

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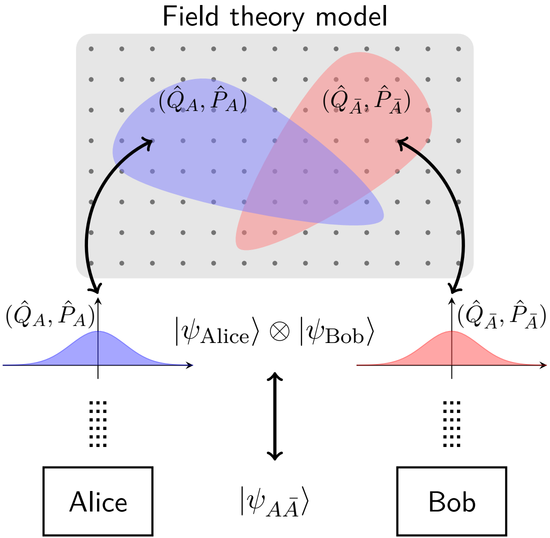

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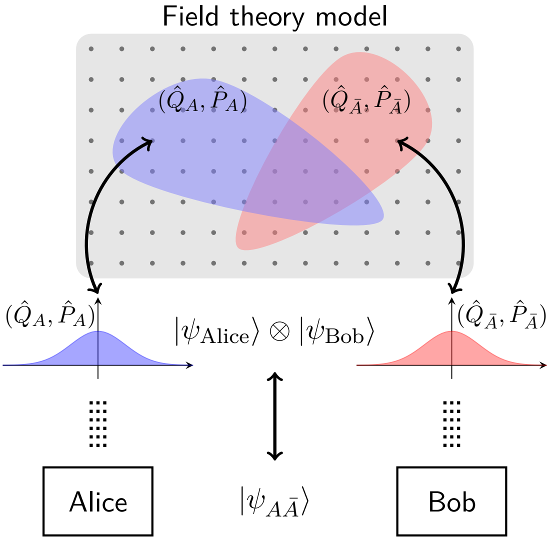

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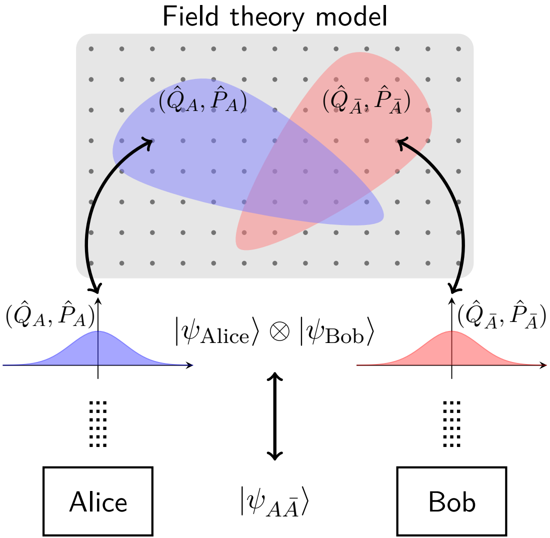

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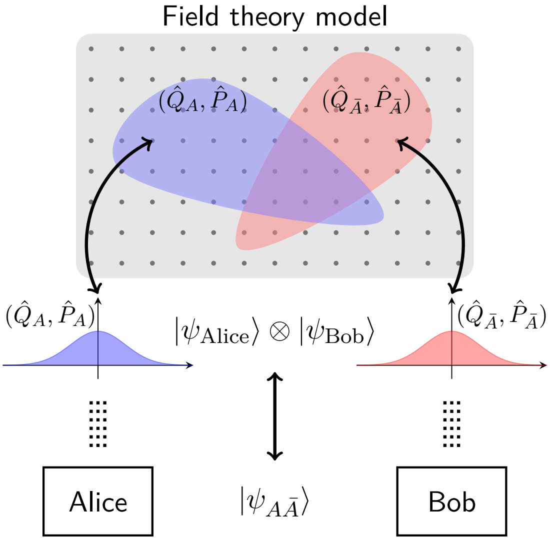

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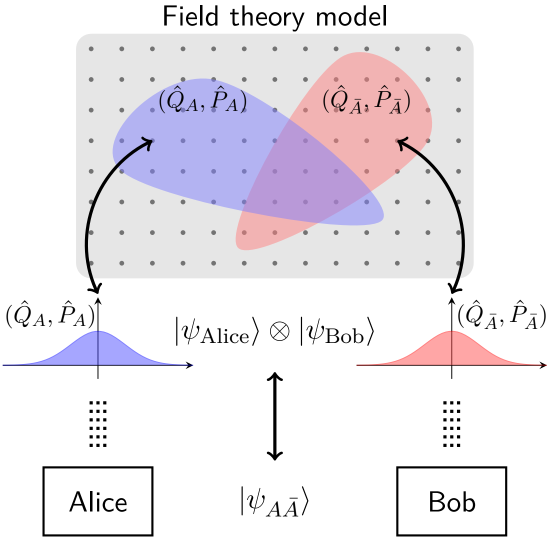
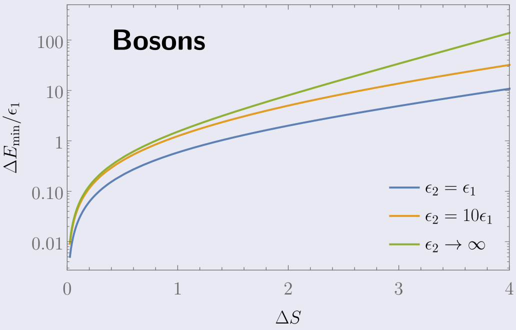
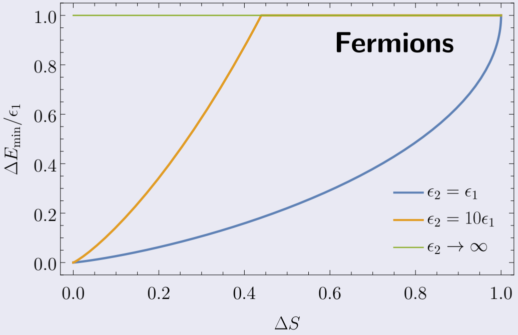
What is the minimal ΔE
to extract a fixed amount ΔS?



Our result

Hamiltonian $\hat{H} = \int_k d^3k\, \omega_k \hat{n}_k + E_0$ has **minimal energy** cost of entanglement extraction:





- ϵ_1 and ϵ_2 are the smallest excitation energies, *e.g.*, $\epsilon_1 = \epsilon_2 = m$ for most field theories.
- In both cases, we find $\lim_{\Delta S \to 0} \Delta E / \Delta S = 0$, *i.e.*, extracting small amounts ΔS is cheap.
- For bosons, we have $\Delta E \sim \frac{\sqrt{\epsilon_1 \epsilon_2}}{2} \exp(\Delta S)$, while $\Delta E \leq \epsilon_1$ for fermions.

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w.r.t. some basis $\hat{\xi}^a = (\hat{Q}_1, \hat{P}_1, \hat{Q}_2, \hat{P}_2)$. We also have $\min(\omega_k) \leq \epsilon_i \leq \max(\omega_k)$.
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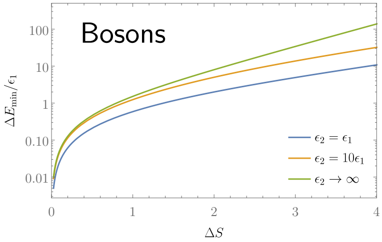
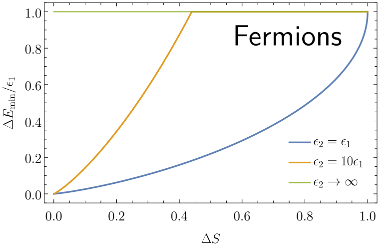
5. **Optimize within all possible choice for $A \oplus \bar{A}$**
   We find that ΔE is monotonic in ϵ_1 and ϵ_2, so choosing them minimally is optimal.
   For quantum fields with mass gap, this implies $\epsilon_1 = \epsilon_2 = m$.

Summary

Summary of key findings:

- Minimal energy cost of entanglement extraction in quadratic systems





- Applies to bosonic and fermionic field theories (and lattice systems)

Outlook for next steps:

- Tighten bounds by including locality assumptions for Alice and Bob.
- Develop perturbation methods to treat interacting systems.